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Some Isomorphisms between Pairs of Latin Squares

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Let A, B, C, D be latin squares with A orthogonal to B and C orthogonal to D. The pair A, B is isomorphic with the pair C, D if the graph of A, B is graph-isomorphic with the graph of C, D. A characterization is given for determining when a pair A, B of latin squares is isomorphic with a self-orthogonal square C and its transpose. Self-orthogonal squares are important because they are both abundant and easy to store. An algorithm either displays a self-orthogonal square C and an isomorphism from A, B to C, C^T or, if none exists, gives a small set of blocks to the existence of such a square isomorphism.

1. DEFINITIONS

A square matrix S of order n with entry $S(i,j)$ in the ith row and jth column is a latin array if $\{S(i,j): 1 \leqslant i, j \leqslant n\} \subseteq N = \{1, 2,..., n\}$. A latin array S is a latin square if for each i and j the sets $\{S(i,j): 1 \leqslant j \leqslant n\}$ and $\{S(i,j): 1 \leqslant i \leqslant n\}$ are both N. A latin array S has the constant lines $Sk = \{(i,j): S(i,j) = k\}$ for $1 \leqslant k \leqslant n$. A family of latin arrays are mutually orthogonal if for each pair A, B from the family the set $\{(A(i,j), B(i,j)): 1 \leqslant i, j \leqslant n\} = \{(i,j): 1 \leqslant i, j \leqslant n\}$. The latin arrays R (for row) and K (for column) are defined by $R(i,j) = i$ and $K(i,j) = j$ for all $1 \leqslant i, j \leqslant n$. Note that if S is a latin array then S is a latin square iff S, R, K are mutually orthogonal. A latin square S is self-orthogonal [1, 2] if S and its transpose S^T are orthogonal. The concept of self-orthogonality is due to Evi Nemeth. Such squares are important because they are abundant and they need only half the storage space of an ordinary pair of latin squares. A latin transversal of a set of latin squares A, B, C,... is a set Δ of cells such that Δ has exactly one cell in each line of each of the families $\{Ri: i \leqslant n\}$, $\{Ki: i \leqslant n\}$, $\{Ai: i \leqslant n\}$, $\{Bi: i \leqslant n\}$, $\{Ci: i \leqslant n\}$,... . Thus, in particular, the set $\{(i, i): i \leqslant n\}$ is a latin transversal of the empty set of latin squares.

If A, B, C,... is a set of $t \geqslant 1$ mutually orthogonal latin squares, then the tactical representation of A, B, C,... is [3, 4] the graph TR(A, B, C,...) with tripartite vertex set consisting of (i) the set X with $t+2$ vertices labeled R, K, A, B, C,..., called square vertices and (ii) the set Y with $n(t+2)$ vertices labeled with the names of the constant lines of R, K, A, B, C,... and (iii) the set Z of n^2 vertices called cells, labeled with $\{(i,j)\colon 1 \leqslant i,j \leqslant n\}$. Each vertex x in X is adjacent with those vertices in Y which bear the names of the constant lines of the square with the same label as x. Thus the square vertex R is adjacent with $R1$, $R2$,..., Rn. Each vertex with the name of a constant line is also adjacent to the n cells having the names of the elements of the constant line. Thus $K2$ is adjacent with $(1, 2)$, $(2, 2)$,..., $(n, 2)$. There are no other adjacencies in TR(A, B, C,...). Since R, K ,A ,B, C,... are all pairwise orthogonal, TR(A, B, C,...) has girth at least 6 and since $t \geqslant 1$, it has girth $\leqslant 6$.

PROPOSITION. *Let G be a tripartite graph of girth $\geqslant 6$ on the vertex set $X+Y+Z$ satisfying*

(i) $|X| = t+2$, $|Y| = n(t+2)$, $|Z| = n^2$;

(ii) *every $y \in Y$ has one neighbor in X and n distinct neighbors in Z.*

(iii) *every $z \in Z$ has $t+2$ neighbors in Y and none in X.*

(iv) *every $x \in X$ has n neighbors in Y.*

Then there exist t mutually orthogonal latin squares A, B, C,... such that $G =$ TR(A, B, C,...).

Proof. Let G satisfy the conditions. Pick any two vertices in X to be labeled R, K and label the rest A, B, C,... . Label the neighbors of R with $R1$, $R2$,..., Rn in any order and those of K with $K1$, $K2$,..., Kn in any order. Since G has girth 6, no two of the Ri have any neighbors in Z in common, nor have any two of the Kj. Moreover, if Ri and Kj are both adjacent to both $z \in Z$ and $w \in Z$ then $w = z$. Thus it is clear that for each $z \in Z$, since z adjoins some Ri and some Kj, that i and j are uniquely determined by z (and of course by our initial labelings of Ri and Kj, but these are fixed), so z may be labeled with (i,j), and if $z \neq w$ then z and w receive different labels. Label the neighbors of A, B, C,... with constant line labels $A1$, $A2$,... . To form the latin square A, say, label the (i,j) entry of a square A with the number k where the cell $(i,j) \in Z$ has the neighbor Ak in Y. It is easy to see that A, B, C,... are mutually orthogonal latin squares and that TR(A, B, C,...) $= G$.

2. THE CASE $t = 2$

Suppose now that the girth 6 graph G satisfies conditions (i)–(iv) with $t = 2$. There are $4 \cdot 3 \cdot (n!)^4$ ways of coordinatizing G as $\mathrm{TR}(A, B)$. Question: When can one be sure that at least one of these is $\mathrm{TR}(S, S^T)$? The following theorem provides a first answer.

THEOREM 1. *If G has girth 6 and satisfies conditions* (i)–(iv) *with $t = 2$, then $G \doteq \mathrm{TR}(S, S^T)$ iff G has a graph automorphism f which acts like a pair of transpositions on X.*

Proof. Suppose $G = \mathrm{TR}(S, S^T)$, relative to some coordinatization of G, then since S and S^T are orthogonal, we define a mapping f as follows:

(i) restricted to $X, f = (R, K)(S, S^T)$;

(ii) on Y, f is the product of all $(Ri, Ki)(Si, S^Ti)$, $1 \leqslant i \leqslant n$;

(iii) on Z, f is the product of all $((i, j), (j, i))$, $1 \leqslant i, j \leqslant n$.

It is easy to see that f is a graph automorphism. Now suppose that f is a pair of transpositions on X. We may label any vertex of X with R, then fR with K. Either remaining vertex can be labeled S and the other T. Label the neighbors of R with $R1, R2,..., Rn$ and label fRi with Ki. For each i we know Ri and Ki have a unique common neighbor xi in Z. Since $Ri \sim xi \sim Ki$

$$\text{iff} \qquad Ki = fRi \sim fxi \sim fKi = Ri,$$

it is clear that f fixes $x1,..., xn$ pointwise. Since $fS = T$, no two of these x's may be in the same line of S or the same line of T, so $\Delta = \{x1, x2,..., xn\}$ is a latin transversal of the graph. Label the lines of S and T so that $xi \sim Si$ and Ti. Label the cells of Z so that $(i, j) \sim Ri$ and Kj, then the set of adjacencies

$$Ri \sim (i, j) \sim Kj$$

implies

$$Ki = fRi \sim f(i, j) \sim fKj = Rj$$

whence

$$f(i, j) = (j, i).$$

Moreover

$$S \sim Si \sim xi \sim Ti \sim T$$

implies

$$T = fS \sim fSi \sim fxi = xi \sim fTi \sim fT = S$$

whence

$$fSi = Ti \qquad \text{and} \qquad fTi = Si.$$

But then

$$\begin{aligned} & S(i,j) = k \\ \text{iff} \quad & (i,j) \sim Sk \\ \text{iff} \quad & f(i,j) = (j,i) \sim fSk = Tk \\ \text{iff} \quad & T(j,i) = k. \end{aligned}$$

This is just the statement that $T = S^T$, so the proof is complete.

If Δ is a latin transversal for G, then there is a particularly nice coordinatization for G obtained as follows. Let $\Delta = \{x1, x2,..., xn\}$ in some order, then label the lines of G so that xi is adjacent with Ai, Bi, Ci and Di for $1 \leqslant i \leqslant n$. Then the cells are given ordered quadruples as labels; if y is in $Ay1$, $By2$, $Cy3$ and $Dy4$ then $y = (y1, y2, y3, y4)$. In particular, $xi = (i, i, i, i)$.

THEOREM 2. *G has an automorphism f which is a pair of transpositions on X iff G does not have 3 (not necessarily distinct) cells y, z, w such that*

$$(y2, y1, y4, y3), \qquad (z3, z4, z1, z2), \qquad (w4, w3, w2, w1)$$

all fail to be cells of G, relative to the coordinatization by the fixed points of f.

Proof. By the proof to Theorem 1, f must take each ith constant line to some ith constant line. Now X has only 3 possible mappings, namely, $a = (AB)(CD)$, $b = (AC)(BD)$, and $c = (AD)(BC)$. If one of these three, say a, extends to an automorphism of G, then it does so uniquely, namely, $a(Ei) = (aE)i$ for $E = A, B, C, D$ and $a(y) = a(y1, y2, y3, y4) = (y2, y1, y4, y3)$. But if a is well defined on Z, that is if each $a(y) \in Z$, then a is indeed a graph automorphism since all adjacencies are preserved.

Remark. No efficient technique is currently known for determining if even one latin transversal exists in such a G [5, Chap. 9], much less for obtaining all of them. To determine that G does not have a latin transversal entails examining all $n!$ permutations of N, even at best, and there is no good characterization which permits one to exhibit a brief proof that such a transversal does not exist. However, this theorem cuts the answer to the self-orthogonality question down to examining the permutations of N

for all possible latin transversals and checking 3 cases for each transversal found. It permits a very brief proof that no automorphism exists for a given transversal, just by exhibiting the three vertices, and if an automorphism does exist for a transversal it is determined by the procedure so that the self-orthogonal square can be exhibited immediately. Unfortunately, very little is known about the automorphisms of tactical configurations other than BIBD. Indeed, it has only recently been discovered by Parker [6] that TR(S, T) can exist without some nontrivial automorphism f of TR(S) which fixes R, K, S pointwise.

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